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Dynamical Critical Exponent for the Majority-Vote Model

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We investigate the dynamical behavior of the isotropic majority-vote model on a square lattice using a combination of damage spreading and finite-size scaling methods. For initial damage $D(0) \geqslant 1/2$, the dynamical phase diagram exhibits a chaotic-frozen phase transition at a critical noise parameter $q_c = 0.0818 \pm 0.0002$, while for $D(0) < 1/2$ the damage does not propagate for any value of the model's parameter $0 \leqslant q < 1/2$. From simulations at q_c, we find that the dynamical critical exponent is $z = 0.65 \pm 0.05$.

KEY WORDS: Damage spreading simulation; majority-vote model; nonequilibrium stationary states; phase transitions, dynamical critical exponent.

1. INTRODUCTION

The damage spreading problem consists in studying the temporal evolution of two initially different configurations of a given system submitted to the same dynamics. Damage spreading simulation together with finite-size scaling theory is a powerful method to study dynamical phase transitions in both probabilistic and deterministic cellular automata,[(1–5)] Ising,[(6–16)] and Potts[(17, 18)] models, spin-glasses,[(19, 20)] and other spin models. For the case of equilibrium spin systems, the damage spreading has been widely investigated by imposing different initial conditions and using specific dynamic rules. These studies revealed the resemblance among the damage spreading and some thermodynamical properties of the system; e.g., the magnetization and the transition temperature.

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In this paper, we address the issue of damage spreading in the isotropic majority-vote model. From the analysis of how the damage between two nonequilibrium stationary configurations submitted to isotropic majority-vote rules evolves in time, for varying amount of damage sites at the start time, we obtain the dynamical phase diagram in the whole parameter space. We also investigate the critical properties of this nonequilibrium model and obtain estimates for the static exponent β/ν and for the dynamic critical exponent z.

We consider the two-dimensional isotropic majority-vote model[21] on a square lattice in which each site is occupied by a spin variable $\sigma_i = \pm 1$. At each time step a randomly chosen spin adopts the sign of the majority of its nearest neighbors with probability p, and the sign of the minority with probability $q = 1 - p$. In other words, the central spin flips with probability q if it agrees with the majority sign of its neighborhood and flips with probability p if it does not. Note that, in this model, the new value of the central spin at time $t+1$ depends on the values at time t of both the spin itself and its nearest neighbors.

In terms of the probability q, which is also called the noise parameter, the flipping probability is defined by

$$w(\sigma_i) = \frac{1}{2}\left[1 - (1-2q)\,\sigma_i S\left(\sum_{\delta=1}^{z} \sigma_{i+\delta}\right)\right], \tag{1}$$

where $S(x) = sign(x)$ for $x \neq 0$, $S(0) = 0$, and the summation is over the z nearest neighbors of the spin at site i. For the square lattice, the neighborhood of a site consists of its four nearest neighbors. The probability given by Eq. (1) exhibits "up-down" symmetry, that is, $w(\sigma_i) = w(-\sigma_i)$ under the change of states of the Ising spins in the neighborhood of σ_i. Moreover, due to the symmetry $w(\sigma_i) = w(-\sigma_i)$ as the model's parameter q goes into $1-q$, we need to consider only the values of q in the interval between 0 and 0.5.

Previous Monte Carlo simulations[22–24] showed that the majority-vote model presents a phase transition to a disordered state at a critical value of the noise parameter, q_c, which depends on the lattice topology. Further, the corresponding critical phenomenon is in the same class of universitily of the equilibrium Ising model,[25] with critical exponents that depend only on the lattice dimensionality. Here we perform damage spreading simulations on the majority-vote model in square lattices of $N = L^2$ sites, for several values of system sizes ranging from $L = 10$ to $L = 120$, with periodic boundary conditions. Section 2 describes the computational procedure, in Section 3 we present our results for the phase diagram and for the dynamical critical exponent. Then, we conclude in Section 4.

2. METHOD

Given a certain configuration at time t, the configuration at $t+1$ was obtained according to the following procedure:

(a) We choose a site at random, site i, and generate a random number r_i uniformly distributed between zero and unity.

(b) If r_i is less or equal to $w(\sigma_i)$, then the chosen spin σ_i is flipped, otherwise it remains in its state.

(c) The above steps is then repeated N times, such that each spin has attempted, on the average, one flip and Monte Carlo time is incremented by one unit.

To study the damage propagation, we start with a random configuration A of spins and leave it evolve in phase space according to the majority-vote dynamics. Once this configuration reachs a steady state, a second configuration B is then created with a fraction $D(0)$ of damaged sites, as compared with those corresponding sites of A. So, we follow the temporal evolution of both configurations until the damage has been relax. In order to assure that both systems evolve within the same dynamics, at each Monte Carlo step (MCS) the same sequence of random numbers is used to update corresponding spins in both configurations. Elapsed the necessary time for the relaxation of the damage, we start calculating the Hamming distance between the configurations, or damage, given by the following time-average:

$$D(t)=\frac{1}{2N}\sum_{i=1}^{N}|\sigma_i^A(t)-\sigma_i^B(t)|. \tag{2}$$

The above quantity is usually dependent on the system size $L=\sqrt{N}$, where N is the total number of spins, on the time of observation t, on the pair of configurations considered, and on the values of the initial damage $D(0)$ and of the noise parameter q. The final damage $\langle D(t)\rangle$ is given by the average over several samples, that is by averaging $D(t)$ obtained with different pairs of configurations and sequences of random numbers.

3. RESULTS

In the following we present our numerical results of damage spreading in the isotropic majority-vote model. We simulate square lattices of linear sizes $L=10, 20, 40, 60, 80, 100, 120$, and with periodic boundary conditions. Starting stationary configurations of the system $(\sigma_i^A(0))$ are obtained after 4000 MCS and the updating is done in a random sequence, i.e., at each step, one of the sites of the lattice is randomly chosen. We then make a copy ($\sigma_i^B(0)$) with a fraction $D(0) = M/N$ of damaged sites. After allowing the damage to relax over a time of 3000 MCS, we measure, over 4000 MCS, its time-average value $D(t)$, given by Eq. (2). For each value of the noise parameter q and initial damage $D(0)$, this procedure is repeated for at least 30 different samples (initial configurations A and B) in order to calculate the mean value $\langle D(t) \rangle$.

Figure 1 shows the average damage $\langle D(t) \rangle$ as a function of the noise parameter q, obtained from simulations on lattices of size $L = 40$, and for values of initial damage $D(0) = 0.60, 0.75, 1.0$. The inset shows the corresponding results for the survival probability $P(t)$, defined as the fraction of samples in which the damage is still propagating at time t. In calculating the average damage for a given q, L and $D(0)$, only these survival events contributed to the average process.

The same numerical results are obtained for other values of $D(0) \geqslant 1/2$, while for $D(0) < 1/2$ the damage does not propagate in the

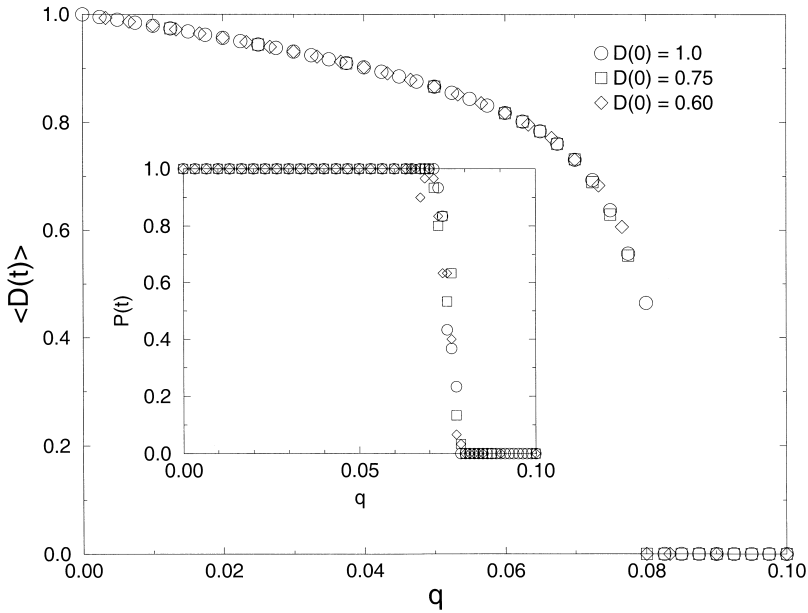

Fig. 1. The average damage $\langle D(t) \rangle$ as a function of the noise parameter q, obtained from simulations on lattices of size $L = 40$, and for values of initial damage $D(0) = 0.60, 0.75, 1.0$. In the inset we plot the corresponding results for the survival probability $P(t)$. We used a total of $t = 11000$ MCS and 30 samples.

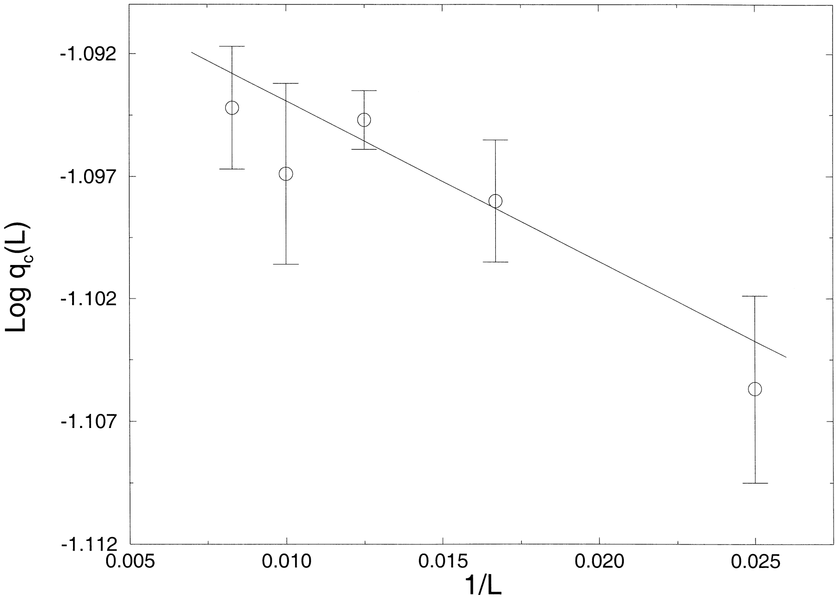

Fig. 2. The size dependence of $q_c(L)$. Data from simulations on square lattices of linear sizes $L = 40, 60, 80, 100, 120$, and for the case of symmetrical initial damage $D(0) = 1.0$.

entire range of $0 \leqslant q < 1/2$. The results presented in Fig. 1 clearly suggests a dynamical phase transition separating a chaotic phase for $q < q_c(L)$, where the damage spreads and the two configurations remain different, and a frozen phase for $q \geqslant q_c(L)$, where the damage heals and the configurations meet in phase space. We recall that, for a finite system the transition becomes a cross over and the present estimates for $q_c(L)$ correspond to the values of the noise parameter above which we obtain a vanishing Hamming distance. For $L = 40$ we obtained the estimate $q_c(L) = 0.0784 \pm 0.0003$. Referring to the work of de Oliveira,[22] here we could mention that the chaotic phase corresponds to the region of nonzero magnetization, while in the frozen phase the magnetization bahaves as $1/L$ and vanishes in the limit $L \to \infty$.

In order to study the L-dependence of $q_c(L)$, we carried out simulations on square lattices of linear sizes $L = 40$, 60, 80, 100, and 120. In Fig. 2 we have plotted $\log q_c(L)$ as a function of $1/L$, for the case of symmetrical initial damage $D(0) = 1.0$. The data are consistent with the finite-size-scaling relation

$$q_c(L) = q_c(1 + bL^{-1/\nu}), \tag{3}$$

where $b = \mathcal{O}(1)$ and we have used the exact value $\nu = 1$ for the static correlation length critical exponent. The best linear fit to the data points yields $b = -0.728 \pm 0.008$ and the following estimate for the critical noise,

$$q_c = 0.0818 \pm 0.0002, \tag{4}$$

which is valid in the limit $L \to \infty$. The above value for q_c is to be compared with $q_c = 0.075 \pm 0.001$ (ref. 22), obtained from Monte Carlo simulations and finite-size-scaling theory.

To determine the dynamical critical exponent for the majority-vote model we have performed simulations at the critical noise parameter $q_c = 0.0818$, considering the case of symmetric initial damage, $D(0) = 1.0$, and for several values of system sizes L. Here we note that, at $q = q_c$, the damage heals within a relatively short time after it is created, and we shall denote the creation time as $t = 0$. In Fig. 3 we have plotted the survival probability, at $q_c = 0.0818$, as a function of time. These data were obtained from averages over 4000, 3000 and 2000 samples for system sizes $L = 10$, 20 and 40, respectively. In Fig. 4 we show the average survival time $T_s(q_c)$ as a

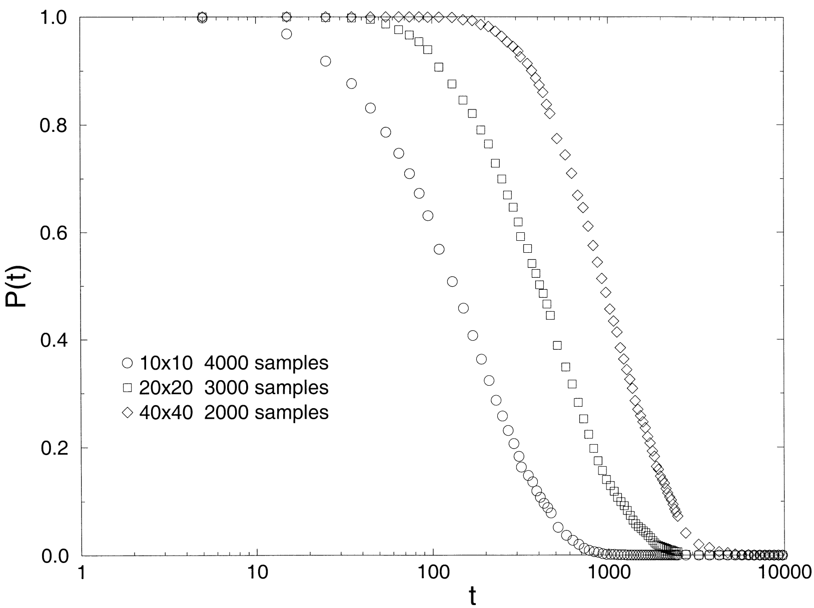

Fig. 3. The survival probability as a function of time, at the critical value of the noise parameter $q_c = 0.0818$. These data were obtained from averages over 4000, 3000 and 2000 samples for system sizes $L = 10$, 20 and 40, respectively.

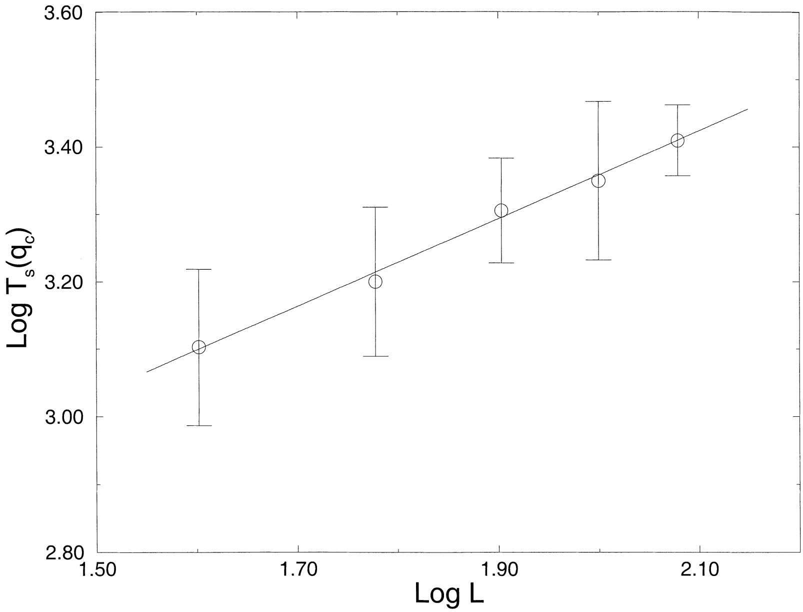

Fig. 4. log–log plot of the average survival time $T_s(q_c)$ as a function of L, for symmetrical initial damage and $L = 40$, 60, 80, 100, 120.

function of L, calculated for the case of $D(0) = 1.0$, and $L = 40$, 60, 80, 100 and 120. Considering a power-law behavior of the form

$$T_s(q_c) \sim L^z, \tag{5}$$

we obtain the dynamical critical exponent $z = 0.65 \pm 0.03$, as the slope of the straight line fitted to the data points in a log–log plot as indicated in Fig. 4.

Figure 5 shows for different values of L, the time dependence of the average damage, $\langle D(t) \rangle$, calculated at the critical value of $q_c = 0.0818$ and considering $D(0) = 1$. The data were obtained from averages over 4000, 3000 and 2000 samples for system sizes $L = 20$, 30 and 40, respectively. In Fig. 6 we have scaled the data of Fig. 5, assuming the following finite-size scaling *ansatz* for $\langle D(t) \rangle$

$$\langle D(t) \rangle \sim L^{-a} F(t/L^z), \tag{6}$$

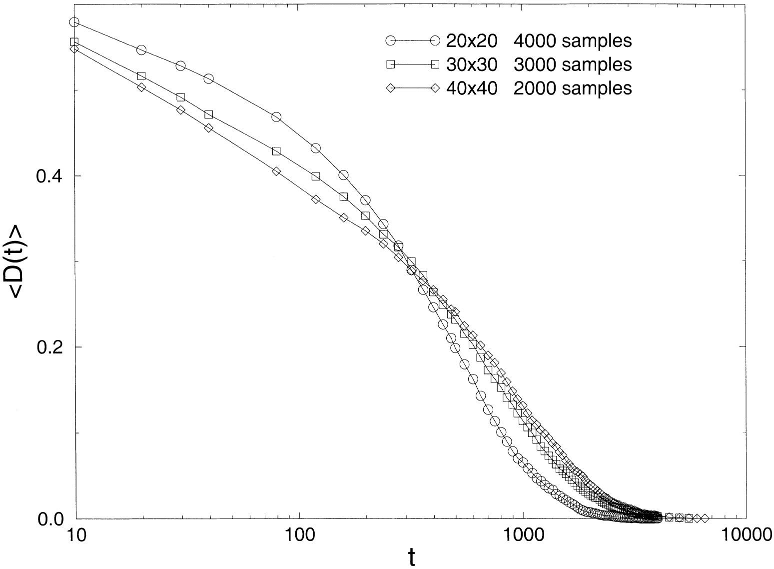

Fig. 5. The time dependence of the average damage, $\langle D(t) \rangle$, calculated at the critical value of $q_c = 0.0818$ and considering $D(0) = 1$. The data were obtained from averages over 4000, 3000 and 2000 samples for system sizes $L = 20$, 30 and 40, respectively.

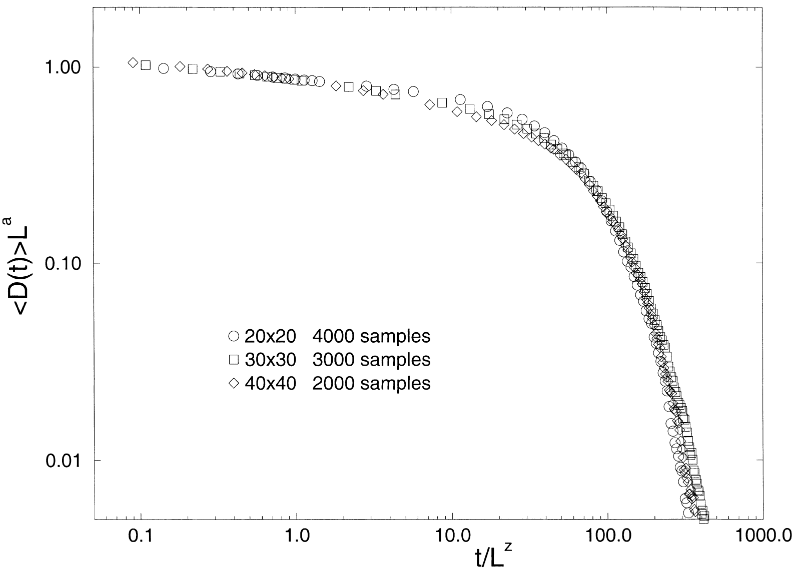

Fig. 6. Data collapse of the results shown in Fig. 5, using the scaling form given by Eq. (6), with $a = 0.125 \pm 0.005$ and $z = 0.65 \pm 0.05$.

where the exponents a and z are lattice independent, whereas the scaling function $F(x)$ depends on the scaled variable (t/L^z) only. The data collapse is obtained with

$$a = 0.125 \pm 0.005 \qquad \text{and} \qquad z = 0.65 \pm 0.05. \tag{7}$$

The above estimates were obtained by assuming the central value of our first estimation [see Eq. (5)] for the dynamical critical exponent, i.e., $z = 0.65$, and then both exponents a and z were varied until the data for lattice sizes $L = 20$, 30 and 40 fall on a single smooth curve. The exponent a is here identified with the ratio β/ν: Our simulated result compares well with previous estimates[22, 23] and with the exact value $\beta/\nu = \frac{1}{8}$ for the two-dimensional equilibrium Ising model. On the other hand, the present result for the exponent z is the first estimate for the dynamical critical exponent of the isotropic majority-vote model, within the context of damage spreading simulation.

4. DISCUSSION AND CONCLUSION

We have considered damage spreading simulation in the isotropic majority-vote model on square lattices with periodic boundary conditions. Depending on the value of the noise parameter q, the resulting phase diagram in the case of initial damage $D(0) \geqslant 1/2$ presents two phases; a chaotic phase in which damage spreads for $q < q_c$, and a frozen phase where damage does not spread for $q \geqslant q_c$. A finite-size scaling analysis yielded for the critical noise parameter $q_c = 0.0818 \pm 0.0002$, in the limit $L \to \infty$, which is in good agreement with the critical value of q_c separating the regions of nonzero magnetization at low q and vanishing magnetization for high q. For $D(0) < 1/2$ the damage does not propagate in the entire range of $0 \leqslant q < 1/2$.

We have also explored finite-size relations for the average survival time and the damage, to determine, from simulations at $q = q_c$, the dynamical critical exponent z and the ratio β/ν. The present estimate of $\beta/\nu = 0.125 \pm 0.005$ agrees with other numerical simulations and supports the symmetry argument of Grinstein *et al.*[25] according to which the present studied nonequilibrium majority-vote model and the equilibrium two-dimensional Ising model are in the same universality class. We mention that the short-time dynamics of the two-dimensional isotropic majority-vote model, and other related models with irreversible dynamic rules having up-down symmetry, has been considered.[26, 27] These works extended the Grinstein's argument to include the dynamical exponent governing the *short-time* relaxational behavior of the magnetization.[28]

Within the context of damage spreading simulation however, we should expect a different picture. In fact, the dynamical exponent for Glauber dynamics has already been calculated by Wang and Suzuki.[13] They considered damage spreading (DS), in $d = 2$, and obtained the estimates $z = 1.3$ with Glauber dynamics and $z = 2.16$ with heat bath dynamics. Moreover, Manna,[14] using DS with Metropolis dynamics found $z = 1.2$, in two dimensions. Accordingly, our calculated value of $z = 0.65 \pm 0.05$ for the two-dimensional majority-vote model yields additional evidence of dynamic rules dependent critical exponents. The observation that the dynamical behavior of model systems, like the Ising model, is strongly dependent on the dynamics used in DS simulations has been widely reported in what concerns phase diagram calculations. The present work and the ones in refs. 13 and 14 show that this is also true for dynamic exponents.

In summary, we have reported damage spreading simulations on the 2D isotropic majority-vote model. The phase diagram of Fig. 1 shows only a weak dependence on the initial damage, whereas the dynamical exponent z determining the power-law growth of the average survival time with system-size [Eq. (5)], is significantly lower than the related exponents associated to the dynamics of Metropolis, Glauber and heat-bath, commonly employed to simulate the equilibrium Ising model.

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